On an Inverse Semi-Group which is Simple but Not Completely Simple

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Abstract

W. D. Munn, (1966) has shown that if E is a semilattice, then the Munn semigroup T_E of E is an inverse semigroup whose semilattice of idempotents is isomorphic to E. If E is a uniform semillatice, then T_E is a bisimple inverse semigroup. J. M. Howie, (1976) proved that up to isomorphism, the only fundamental bisimple semigroup S, having

$$E = C_W = \{ e_0, e_1, e_2, \ldots\ldots\ldots\ldots\ldots\} \text{ with } e_0 > e_1 > e_2$$

Is the Bisyclic semigroup. In this paper is highlighted the structure of the Bicyclic semigroup and prove that every simple semigroup which has idempotent, none of which is primitive, must necessarily be a web of bicyclic semigroups.

Keywords

Bicycle Semi-Group Structure; Inverse Semi-Group; Simple Semi-Group; Bisimple Semi-Group; Completely Simple Semi-Group.

Preliminaries

Construction

Let X be a 2- element set, i.e. X = { p,q}. We construct a semigroup B, from X by letting all elements in B, be of the form $q^m p^n$, m and n are nonnegative integers. We also let $pq = p^0 = q^0 = 1$. Then each element $q^m p^m \in B$ is unique. It is trivial to show that B is indeed a semigroup under the composition, multiplication.

Definitions

1 Let *a, b* $\in$ *S*, where *S* is a semigroup. Then a and b are (generalized) inverse of each other if *aba = a and bab = b*. S is called an inverse semigroup if every element of S has a unique inverse in S.

2 A semigroup *S*, is simple if S has no proper ideals

3 For every element *a, b* is a semi group *S*,

$$a \,\&\, b\,[a\,\Re b] \Rightarrow a \cup S_a = b \cup S_b [a \cup aS = bS]$$
$$\text{Also, } a \quad b \Rightarrow a\ L\ o\ \Re b \Rightarrow a\Re o L b$$
$$a\hbar b \Rightarrow a\ \ L\ \ b\ \text{ and } b\ \ \Re a,\ ,\ \text{i.e } H = L \cap \Re$$

4 A semigroup S, is bisimple (i.e. D – simple) if S consist of a simple D-class.

5 A completely simple semigroup is a semigroup which contains a primitive idempotent.

Propositions

1. The semigroup B, constructed in section 1.1 above is the Bicyclic Semigroup. It is an inverse semigroup which is bisimple (hence simple), with an identity element. The idempotent of B are of the form.

$$e_n = q^m p^n \in B(m, n = 0,1,2,3....) \quad \text{which satisfy}\,; 1 = e_o > e_1 > e_2, > e_3 ... e_n > e_n + 1...$$

And so B has no primitive idempotent i.e B is not completely simple (Prop. 1)

2. Every simple semigroup which has idempotents, none of which is primitive, must contain B. Such a seigroup has not be a “web of B’ s” (Prop. 2)

Proof of Propositions

Proof of Prop 1

1. B is an inverse Semigroup

$$e_n = q^m p^n \in B(m, n = 0,1,2,3....)$$
$$B = q^m p^n \in B, \text{so that } aba = q^m p^n (p^n q^m) q^m p^n$$
$$\text{i.e } aba = (q^m p^n q^n p^m) q^m p^n) = q^m p^m q^m p^n$$
$$= q^m p^n (\text{Since } 1 = p^m q^m = 1 \in B$$
$$= a \in B$$
$$\text{also, } bab = q^n p^m (q^m p^n) q^n p^m = q^n p^m = b \in B$$

Hence B is an inverse semigroup.

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2. B is a Bisimle Semigroup

Let $a = q^m p^n \in B$ and $b = q^r p^S \in B (m, n, r, s = 0,1,2,3,...)$

We now show that a $\Re$ b [a L b]

i.e. aB = bB = B [Ba = Bb = B]

Since $q^o p^o = 1.1 = 1 = pq = 1 \in B$

$aB = \{q^m p^n q^1 : m, n, l, k = 0,1,2,3,....) = \{q^{m+1-np} k; if 1 \geq n\} = \{q^{m+1-np} k; if 1 \geq n\}$
$(since\ pq = 1 \in B)$

Similarly,

$bB = \{q^r q^s q^1 p^k ; r, s, l, k = 0,1,2,3,....)$

$\{q^{r+1-s} p^k ; if\ \ 1 \geq s\}$

$\{q^r p^{k+s-1} ; if\ \ 1 \geq 1\}$

We cannot compare aB and bB under these conditions. So we let $q^m p^n = [m, n]$ and let min [m,n] = (m,n) and define

$$q^m p^n q^r p^s = [m, n][r, s] = m + r - (n, r); n + s\ \ (n, r)]$$

Then let $a = [m, n]$ and fix $b = [m, s]$ which is possible

$\sin ce\ \ b = qrps \in B : s, r = 0,1,2,....n...$

So,

$[m, n][m, s] = [m + n - (n, m), n + s - (n, m)] = [m, s] \in aB$

Similarly,

$[m, s] \in bB. hence, aB = bB$

i.e. a $\Re$ b

Conversely; suppose that [m,n] $\Re$ [r,s], then there exist

$[x_1, y_1], [x_2, y_2] \in B$ such that :

$[m, n] = [r, s][x_1, y_1] = [r + x, - (s, x), s + y_1\ \ (s, x_1)]$

and

$[r, s] = [m, n][x_2, y_2] = [m + x_2, - (n, x_2), n + y_2\ \ (n, x_2)]$

$hence, m = r + x_1\ \ (s, x_1)$

$r = m + x_2\ \ (n, x_2)$

If $m = r$, then we must have $x_1 \leq s$ and $x_2 \leq n$

Suppose that $m \neq r$, then $x_1 > s$ and $x_2 > n$

So, $m = r + x_1 - s$ i.e. $m \geq r$

$r = m + x_2 - n$, i.e. $r \geq m$

hence, $m = r$

Thus, $[m,n]\ \Re\ a\ [r,s] \Leftrightarrow m = r$

So the R - classes of $a \in B = q^m p^n = \{q^m p^x, x = 0,1,2,3,\ldots\ldots\ldots\}$

Similarly, we can construct the arg ument to whos that in B, $Ba = Bb$ i.e.

a L b, for every [m,n] L [r,s]

i.e the L classes of a $q^m p^n = \{q^r p^n, y = 0,1,2,3,\ldots\ldots\ldots\}$

Thus, for every $[m,n],[r,s] \in B; [m,n]\Re[m,s]$

and $[m,s]\,L\,[r,s]$. Hence B is Bisimple.

Existence and Nature of Idempotents in B

Let $(q^m p^n)^2 = q^m p^n \in B, q^m p^n q^m p^n = q^m p^n$

Then, $[m,n][m,n] = [m + m - (n,m), n + n\ (n,m)] = [m,n]$

i.e. $m = m + m - (n,m) \Rightarrow = m$ (i)

$n = n + n - (n,m) \Rightarrow = n$ (ii)

from (i) and (ii) $\Rightarrow m = n = (n,m)$

Thus the idempotents of B are of the form $e_n = q^n p^n$ $(n = 0,1,2,3,\ldots\ldots)$

Now $e_n \geq e_{n+1}$ for every $n \in N$

proof :

$e_n e_{n+1} = qnpn(qn + 1pn + 1) = qn(pnqn)q1pn + 1 = qn + 1$

$= en + 1.$thus $e_n e_{n+1} = e_{n+1} e_n \neq e_n$

i.e. $e_0 = 1 > e_1 > e_2 > e_3 > \ldots\ldots\ldots\ldots > e_n\ e_{n+1}$

Thus B has no primitive idempotent

$\Rightarrow$ B is a Bisimple semigroup $\Rightarrow$ B is a simple semigroup

$\Rightarrow$ B is not completely simple.

The H – classes of B and the Left [Right] ideals of B

For $a = qmpn \in B$, the L - Classes of a,

$La = \{q^y p^n; y = 0,1,2,3,....\}$ and the R - classes of a

$Ra = \{q^m p^x; x = 0,1,2,3,....\}$ also, B is D - simple

now $a\, \mathsf{H}\, b \Rightarrow [r,s]\ [m,n]\, \mathsf{L}\, [r,s]$ adn $[r,s]\, \mathsf{R}\, [m,n]$

$\Rightarrow s = n$ and $r = m$,

$\Rightarrow a = b$

Hence the H-class of B are the singletons $[m,n], m,n \in \{N \cup \{0\}\}$

Left Ideals: Let L be a left ideal of B. Then L = ∪B b, b ∪∈L

Let $b = [r,s] \in L$, then $a = [m,n] \in Bb \Leftrightarrow a = a = [k,l][r,s]$

For some k,l i.e $a = k + r - (1(l,r)]$ is $1 \geq r$,

Then $[m,n] = [k,1\ + s - r] \Rightarrow m = k,\ n = s + 1 - r$, i.e. $n \geq S$

$$So\{[x,y], \left\{ \begin{matrix} x=0,1,2,3,... \\ \\ y=s,s,+1,s+2,.... \end{matrix} \right\} \subseteq R[r,s]$$

If $1 < r$, then, $[m,n] = k + r - 1, s]$i.e. $n = s$

Hence, $[m,n] \leq [x,y];_{y=s,s+1,s+2,....}^{x=0,1,2,3,,....}$

Similarly,

for $b = [r,s] \in B, B[r,s] = [x,y];_{y=s,s+1,s+2,....}^{x=0,1,2,3,,....}$

Hence, if $t > s$; B is $L = \cup B[r,s]$

Now, let a left ideal o fB is $L = \cup B[r,s], [r,s] \in L$

i.e.$L = B[r,s_o]$where $s_o\ = \min\{r,s\}$

i.e the left ideals are :

$B[0,0], B[1,1], B[2,2],......$

And $B[j,j]\ \leq B[i,i]$for $j > i$

Similarly, the right ideals are $[0,0], B[1,1], B[2,2],......$

where $B[k,k]\ \leq B[j,j]$for $k > j$

Proof of Prop. 2

Theorem 1

Let e be a non – primitive idempotent of a simple semigroup S, which is not completely simple. Then S contains a bicyclic subsemigroup which has e as their identity element.

Lemma 1

Let *e, Ia, Ib*, be elements of a semigroup S, such that; ea = ae = a; eb = be = b, ab =e But ba ≠ e. Then every element of the subsemigroup generated by a and b is uniquely expressible as $b^m a^n$ where m,n are nonnegative integers.

And $a^0 = b^0 = e^0$. Hence the subsemigroup of S generated by a and b is isomorphic to the bicyclic semigroup B

Proof of Lemma I

<u>S is infinite</u>: We prove that $a^n \neq a^r$ for $1 \leq r<n$

For all nonnegative integer n.

$$\text{Now } a^2 = a \Rightarrow a2b = b = a(ab) \Rightarrow ae = a = e$$

(which is a contradiction). So, $a^2 \neq a$.

Suppose that a, is such the $a^n \neq a^k$, for $1 \leq k < n$.

Let r be such that $1 \leq r \leq n + 1$ and $a_{n+1} = a^r$

Then $a^n(ab) = a^{r-1}(ab) \Rightarrow a^n e = a^{r-1}e \Rightarrow an = ar - 1$, but $1 \leq r \leq n-1 \Rightarrow 1 \leq r -1 < n$, hence $a^{n+1} \neq a^r$ for $1 \leq r \leq n+1$.

We conclude that $\{a^n; n = 0, 1,2,3,\ldots\ldots\ldots..\}$ is infinite.

So, S is infinite.

Also, $a^n b^m \Rightarrow a^m a^n = a^{mb} m \Rightarrow a^{m+n} = e = a^0$ by the above reasoning. i.e. m + n = 0. But both m and n are nonnegative integer so we must have m = n = 0.

Thus $a^n \neq b^m$ unless m = n = 0.

Now, suppose that; $b^n a^m = b^r a^s$, we may assume that $n \leq r$, without loss of generality. Thus we have;

$$a^m = b^{r-n} a^s \Rightarrow a^{m-s} = b^{r-n}$$
$$\text{so}, m - s = r - n = 0 \Rightarrow m = s, r = n.$$

Hence the expression $b^n a^m$ is unique.

Therefore the subsemigroup generated by {a,b} is isomorphic to the bicyclic semigroup B.

Proof of Theorem I

Let $e \in S$ be a non–primitive idempotent. Then there exits an idempotent $e_1 \in S$, say, such that

$e_1 = ee_1 = e_1 e$ but $e_1 \neq e$

Since S is simple, there exist $x, y \in S$ adn $e = x\, e_1 y$ (because for every $a \in S$, $SaS = S$)

Take $a, b \in S$ such that; $a = e \times e_1 b = e_1 ye$

then $a, b \in S$ such that; $a = e \times e_1 = a$

$ae = (e \times e_1)e = e \times e_1 = e \times e_1 = a$

i.e.$ea = ae = a$

Similarly, $eb = be = b$

aslo, $ab = (e \times e_1)(e_1 ye) = e \times (e_1 e_1) ye = e(xey)eee = e$

Now, if $ba = e$, then $(e_1 ye)(e \times e_1) = e \Rightarrow e_1 yexe_1 = e \Rightarrow (e_1 yexe_1)e_1 = ee_1 = e$

$\Rightarrow e = e_1$ (which is a contradiction!!) since $e \neq e$.

So $ba \neq e$. Hence the conditions in Lemma I are satisfied. So the bisyclic subsemigroup generated by {a,b} is a subsemigroup of S containing e as an identity element.

Conclusion

From theorem I and the structure of the bicyclic semigroup *B*, thus discussed, it is evident that any semigroup *S*, which is simple, and which contains a non-primitive idempotent which acts as an identity element, must comprise *a* "*web of B*'s."

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