Issue 11, July-December 2007
p. 61-68

On Primitive Abundant Semigroups and PA-Blocked Rees Matrix Semigroups

Lawrence N. EZEAKO

Department of Mathematics and Computer Science, Federal Univesity of Technoly, Minna, Nigeria
abochoche@gmail.com

Abstract

In this paper we utilize Warne’s homomorphism theorem for bisimple inverse semigroups with an identity element [13, Theorem 1.1] to prove the theorem: Every primitive abundant semigroup with a zero element is isormorphic to a PA-blocked Ree’s matrix semigroup.

Our approach obviates the tedium of the proof of the existence of a homomorphic mapping (see Markie, 1975) in order to prove the same theorem.

Keywords

Bisimple; Inverse; Semigroups; Primitive Abundant Semigroups; Ree’s Matrix Semigroups; Homomorphism

Preliminaries

Ree’s matrix Semigroup (A completely O-Simple Semigroup)

Let $G \equiv$ a simple group with identity e $\left.\begin{matrix} I \\ \wedge \end{matrix}\right\} \equiv \text{non} - \text{empty sets}$

P $\equiv$ matrix (pμ with entries from the zero group G^0, where

$G^o \equiv (Gu\{0\}) i.ea \Lambda \times 1$ Matrix

Let $S \equiv (G \times I \times \Lambda) u \{0\}$

where, $(a, I, \lambda)(b, j, \mu) = \{^{ap\lambda j, b, I, \mu) \text{if } P\lambda j=0}_{oifP\lambda j=0}$

$G^0 \equiv (G\cup\{0\}$ i.e. a $\Lambda \times I$ matrix

Let $S = (G x I x \Lambda) u \{0\}$

$$(a, I, \lambda)(\beta, j, \mu) = \begin{cases} (ap_{\lambda j}b_i, \mu) \text{ if } P_{\lambda j} = 0 \\ 0 \text{ if } P_{\lambda j} = 0 \end{cases}$$

Then S thus constructed is completely O - simple, and $S \equiv M^o\,[G,I,\Lambda,P]$ is the $I \times \Lambda$, Ree's matrix semigroup over the O-group G^o, with the regular sandwich matrix P.

The PA - blocked Ree's Matrix Semigroup

D. Rees [9] has shown that we can construct a semigroup from a set of monoids and bisystems over these monoids. Such a construction generalizes Rees matrix semigroup. Under certain conditions (see [9]) such as semi group becomes abundant, with all its non-zero idempotents primitive. It is called a PA-blocked Rees matrix semmigroup.

Important remarks: Since S is O-Simple

- $\{0\}$ and S are the only ideals of S
- $S \times S \neq 0$
- $\forall a \in s/\{0\}$; aS = Sa = S

Also, since S is completely 0 - simple, S satisfies the Min_L, Min_R conditions i.e every non-empty set of the l - classes or the R-classes possesses a minimal member

Right [Left] S- Systems

If M is a set and S is a monoid then M is a right S-system if there exists a mapping $(x,s) \to xs$ from M x S into M, with the properties that

i. $(x,s)t = x(st); x \in M; s,t \in S$

ii. $x1 = x; x \in M$

a left S - system is dually defined. If S and T are monoids, then M is an (St) - Bisystem, where M is a left S - system as well as a right T - system and for each $s \in S, t \in T, x \in M$.

we have $(x,s)t = s(xt)$

Leonardo Journal of Sciences
ISSN 1583-0233

Issue 11, July-December 2007
p. 61-68

If M is a right S - system and N is a left S - system and τ is an equivalence relation on M x N generated by the subset $\{[(xs, y), (x, sy)]; x \in M, s \in S, y \in N\}$ of $(M \times N)(M \times N)$, then the set $(M \times N)l\tau \equiv M\otimes, N \equiv M \otimes N$, simple, is called the tensor Product over S, of the two S-systems

* using these definition we make the following propositions

Propositions

Let e, f be idempotent of a primitive abundant semigroup S, with zero, if e, f are not D-related and if ef, fe are both non-zero, the;

1. H* ef and H* fe are cancellative subsemigroups of S without identities
2. /H*e/ = /H * ef/ = /H* fe/ = /H* f/
3. H* ef is isomorphic as a semigroup and as a right H* f - system to a left idea of H*f, and isomorphic as a semigroup and as a left H*e - system to a left ideal of H*e

Proof of Proposition 1

$ef \neq 0$ and $ef \in eS \cap Sf$

$\therefore$ by [4 corollary 2], $H^* ef \equiv R^* ef \cap L^* f$

Again $fe \neq 0, \therefore$ [4 lemma 3]

$H^* ef. H^* ef = \{0\} \therefore$ by [4 corollary 3] $H^* ef$ is a cancellative subsemigroup of S

Now, $H^* ef. \equiv R^* ef \cap L^* f \Rightarrow H^* ef$ contains an indempotent g,
say, such that fLg and gRe
i.e. $(f, e) \in L \text{ o } R \Rightarrow fDe$ (contracdiction)

$\therefore H^* ef$ is a cancellative subsemigroup of S without identify so i) is proved

Proof of Proposition 2

If $t \in H^* f$ by [4 Lemma 3] $et \neq 0$ and so by [4 Lemma 2] $et \in R^* e \cap L^* f$

Also, $ft = t$ so that $t_1 t_2 \in H^* f$ and $et_1 = et_2$, then eft_1 eft_2

Since $ef \in L^* f \Rightarrow ft_1 = ft_2$ i.e $t_2 = t_2$

Hence $/ H^* f / \subseteq / H^* ef /$ (a)

Also, if $x \in H^* f$, then $Fx \in H^* f$ and $ex = x$

Thus

If $x_1, x_2 \in H^* f$ and $fx_1 = fs_2$, then

$fex_1 = fex_2$ $ex_1 = ex_2$ since $fe \in L^*e$

$\therefore x_1 = x_2$ i.e. $/H^*f/ \subseteq /H^*ef/$ (b)

From (a) and (b) $\Rightarrow$ $/H^*f/ = /H^* ef/$

Similarly, $/H^* ef/ = H^*el$

and 2 is proved.

Proof of Proposition 3

Let $\Phi : H^* \ ef \rightarrow H^*f$ such that $\forall x\ H^*ef, (x)\ \Phi = fx$, then Φ is injective

Also, $(H^*f)\Phi = \{fx : x \in H^*ef\} \subseteq H^*\ f$ is a right ideal of H^*f

For each $H^*f; (xt) = f(xt) = (fx)t = ((x))t$

$\therefore \Phi$ is an isomorphism from $H^*\ ef$ onto Im.Φ

Furthermore, for any $x, y \in H^*ef, (xy)\Phi = f(xy)$

$= f(xf)y = (fx)fy) = (x)\Phi(y)\Phi$

So, Φ is a semi group isomorphism

Similarly, we can show that H^* ef is isomorphic as

a left H^* e - system to a left ideal H^*e

i.e. 3 is proved.

Construction of a Primitive Abundant Semigroup S, with Zero

Let $I \equiv$ the set of nonzeroR^* - classes of S

Let $\Lambda \equiv$ *the set of nonzero* L^* *- classes of S*

$\therefore R^*$ *- classes* $\equiv L_\lambda (\lambda \in \Lambda)$

for *each* $(i,\lambda) \in I \ x\ \Lambda; H_{i\lambda} = R_1^*\ L_\lambda^*$

Leonardo Journal of Sciences
ISSN 1583-0233

Issue 11, July-December 2007
p. 61-68

$\Rightarrow$ every nonxero H^* - calss of S is some $H^*{}_{i\lambda}$ and each $H^*{}_{i\lambda}$ is either epty or it is an H^* – class

NOTE:$S-\{0\} = \cup \{H^*{}_{i\lambda}$ $(i,\lambda(i \in$ I x ΛI

Now Let r $\equiv$ the set of nonzero D - class i.e. these D_a - classes = D $(a \in r)$

For each $\alpha \in r$, define

$I_a = (i \in I: D_a \cap R_i^* \neq \varphi\varphi$

$A_a = (\lambda \in \Lambda: D_a \cap L_{\lambda i}^* \neq \varphi\varphi$

For each $i \in I$, $i \in I_a$ for some a, since every R^* - class of S contains an idempotent.

*I*n each R^* - class, all the regular elements are R^* related. So an R^* - class has a nonempty intersection, with only one regular D - class. Hence; if

$\alpha\beta \in \Gamma$ with $\alpha \neq \beta$, then

$I_a \cap I_\beta = \varphi$

So that $\{I_\alpha : \alpha \in \Gamma)$ is a partition of Λ

Now, if an H^* - calss contains an udempotent, then there exists $\alpha \in \Gamma$

and (i,λnd$\in$ Γ_α x Λ_α such that $H^*{}_{i(\alpha(\alpha)\lambda(} \cap L^*{}_{\lambda(\alpha)}$

is an H^* - class which contains an idempotent.

Let $H^*{}_{i(\alpha(\alpha)\lambda(} \equiv M_{\alpha\alpha} = T_\alpha$

Then T_α is a monoid i.e. cancellative subsemigroup of S with an identity.

T_α is also indepedent (in structure) of the choice of H^* - class containg an idempotent idexed by a pair in I_α x Λ_α (see [10])

Let e_α $\equiv$ the identity element of T_α for each pair (i,λa $\in$ $I_\alpha \cap \Lambda_\alpha$

from the propositions in 2.0, we have that;

i) $H^*{}_{i\lambda} \neq \varphi$

ii) There exists regular elements $r^\alpha{}_i$, $q^\alpha{}_\lambda$ in $H^*{}_{i(\alpha(\alpha}$, respectively such that

$x \mapsto r_i^\alpha x$ is a bijection from $H^*{}_{i(\lambda(\lambda}$ onto $H^*{}_{i\lambda}$

Thus once we have chosen

iii) $\{r^\alpha{}_i : i \in I_\alpha, \alpha \in \Gamma\}$ and $\{q_\lambda{}^\alpha : \lambda: \in \Lambda_\alpha\ \alpha \in \Gamma\}$

we have a unique expression r^α l $aq^\alpha{}_\lambda$ $(\alpha \in T_\alpha)$ for each element of $H^*{}_{i\lambda}$ where $(i,\lambda) \in I_\alpha \cap \Lambda_\alpha$

Main Theorem

Every primitive Abundant Semigroup as constructed in (3.0) above is Isomorphic to A PA-Block Rrees Matrix Semigroup

Proof of Main Theorem

Now for $\alpha\beta \in \Gamma\Gamma$ with $\alpha \neq \beta$

Let $M_{\alpha\beta} = H^{*}_{i(\alpha(\alpha)\lambda(}$

$H^{*}_{i\lambda} \neq \varphi \Leftrightarrow M_{\alpha\beta} \neq \varphi$

Assume that $M_{\alpha\beta} \neq \varphi\varphi$ then $_{\alpha\beta}$ is a strongly torsion free T_α, T_β - bisystem.

if $(i,\lambda f \in I_\alpha \times \Lambda_\beta$ then

[Proposition 2.0] imply that $x \mapsto xq^{\beta}{}_{\lambda}$ is a bijection from $H^{*}{}_{i(\alpha(\alpha}$ onto $H^{*}{}_{i\lambda}$

Remark that r^{α} i, $q^{\beta}{}_{\lambda}$ are the regular elements of $H^{*}{}_{i(\beta(\beta),}$

respectively which have chosen before.

Hence each element of $H^{*}{}_{i\lambda}$ may be written uniquely as:

$r_i^{\alpha} mq_\lambda^{\beta}$ where $m \in M_{\alpha\beta}$

Also

$H^{*}{}_{i\lambda}$ is a (T_α, T_β) - bisystem under the action

$t_{\alpha}(r_i^{\alpha} mq_\lambda^{\beta}) = r_i^{\alpha} t_\beta mq_\lambda^{\beta}$

(T_α, T_β) is isomorphic to $M_{\alpha\beta}$ and the bisystem structure of $M_{\alpha\beta}$ is idependent of the possible choices of T_α and T_β

Let αet $\alpha\gamma \in \Gamma$ and suppose that $M_{\alpha\beta}$ $M_{\beta\gamma}$ are both non - empty.

Let $a \in M_{\alpha\beta}$ $b \in M_{\alpha\beta}$ then $aL^{*}{}_{e\beta}R^{*}$ b

By [4 Lemma 3] $ab \neq 0$

Since $ab \in aS \cap Sb \subseteq e\alpha e\alpha \cap Se\gamma$

$\Rightarrow e\alpha e^{*}{}_{ab}L^{*}{}_{e\gamma}$from corollary 1

$\therefore ab \in R^{*}{}_{i(\alpha(} \cap L_{\lambda(\gamma)} = M_{\alpha\gamma}$

and $M_{\alpha\beta} \neq \varphi\varphi$

$\varphi_{\alpha\beta\gamma}: M_{\alpha\beta} \otimes M_{\beta\gamma} \mapsto M_{\alpha\gamma}$

Conclusion

We have thus proved in a very simple manner i.e. without recourse to the necessity for the existence of a homomorphic mapping,-that every primitive abundant semigroup with a zero, is isomorphic to a PA - blocked Rees Matrix semigroup

Issue 11, July-December 2007
p. 61-68

Acknowledgements

I sincerely wish to thank Professor K. R. Adeboye, former Dean School of Science and Science Education, Federal University of Technology, Minna, for his useful comments on an earlier draft of this paper.

References

1. Armstrong S., *The Structure of Type A Semigroups*, Semigroup Forum 1984, 29, 319-336, Springer Verlag, New York, Inc.
2. Clifford H., Preston G. B., *The Algebraic Theory of Semigroups*, Math. Surveys, 1961, 7(1) & 1967, II Amer. Math. Soc. Providence, R.I.
3. Fountain J. B., *Adequate Semigroups*, Proc. Edin. Math. Soc., 1979, 22, p. 113-125
4. Fountain J. B., *Adequate Semigroups*, Proc. London Math. Soc., 1982, 44(3), p. 103-129
5. Howie J. M., *An Introduction To Semigroup Theory*, Academic Press, 1976.
6. Markie L., *On Locally Regular Rees Matrix Semigroups*, Acta. Sci. Math (Szeged), 1975, 37, p. 95-102.
7. Munn W. D., *A Class of Irreducible Matrix Representation of an Arbitrary Inverse Semigroup*, Proc. Glasgow Math. Assoc., 1961, 5, p. 41-48.
8. Pastigin F., *A Representation of a Semigroup of Matrices Over a Group with Zero*, Semigroup Forum, 1975, 10, p. 238-249.
9. Rees D., *On Semigroups*, Proc. Cambridge Philos. Soc., 1940, 36, p. 387-400.
10. Rees D., *On The Ideal Structure of a Semigroup Satisfying a Cancellation Law*, Quant. J. Math. Oxford Ser., 1948, 19(2), p. 101-108.
11. Reilly M. R., Clifford A. H., *Bisimple Inverse Semigroup as Semigroups of Ordered Triples*, C and .J. Math., 1968, 20, p. 25-39.
12. Steinfeld O., *On a Generalization of Completely O-Simple Semigroups*, Acta. Sci. Math. (Szeged), 1967, 28, p. 35-46.

13. Warne R. J., *Homomorphisms of α - Simple Inverse Semigroups with Identity Pacific,* J. Math., 1964, 14, p. 1111-1122.