Comparison of Euler and Range-Kutta methods in solving ordinary differential equations of order two and four

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Abstract

The purpose of this to produce efficient numerical methods with the same order of accuracy as that of the main starting values for exact solutions of fourth order differential equation without reducing it to a system of first order differential equations. The methods of the differential systems arising from the approximate solution to the problem are adopted using the Runge-Kutta method and stages. The methods were compared and contrasted based on the results obtained. The comparison shows that Euler method gives accurate approximate result than Runge-Kutta method. After the derivation of the formulae of $O(h^2)$, the comparison was done in regards to identify the formula with higher accuracy.

Keywords

Numerical Analysis; Numerical Approximation; Exact Solution; Accuracy; Runge-Kutta and Euler

Introduction

Numerical analysis is a branch of mathematics that deals with the study of methods and procedures used to obtain approximate solutions to mathematical problems. EndreSull and David Mayer defined the numerical analysis as a branch of mathematics that provides the theoretical foundation for the numerical algorithm we rely on to solve a multitude of computational problem in mathematical models or the study of algorithms that use numerical approximation (as opposed to general symbolic manipulations) for the problems of mathematical analysis [1]. Numerical analysis naturally finds applications in all fields of engineering and the physical science, but in this 21st century, the life science and even the arts have adopted elements of scientific computations [2].

The overall goals of the field of numerical analysis in the design and analysis of techniques to give approximate but accurate solution are hard to get. It is therefore, important to be able to estimate the error involved in such approximation. Thus, the aims of this work was to compare between Euler and Runge-Kutta methods to a rigorous analysis in order to demonstrate the efficiency of the methods to other similar techniques. It was also examine the effect of the steps on the accuracy of the techniques.

Euler’s method is more preferable than Runge-Kutta method because it provides slightly better results. Its major disadvantage is the possibility of having several iterations that result from a round-error in a successive step.

Secularity band differences in the results of some numerical methods with the standard Euler’s method of order three and four was examined.

Material and method

Euler method

In mathematics and computational science, the Euler method is a first-order numerical procedure for solving ordinary differential equation (ODEs) with a given initial value. It is the most basic explicit method of numerical integration of ordinary differential equation and is the simplest Runge-Kutta method. The Euler method is named after Leonhard Euler (1707) [3].

Two approaches named standard Euler method and modified Euler method are known.

Standard Euler method

The standard Eular method which is the first order Runge-Kutta method was derive by Leonarhd Euler (1707-1783) [4].

Consider the initial value problem, the first order

$$y'(x,y) = f(x,y); \quad y(x_0) = y_0 \tag{1}$$

where y' is the first order differential equation; $f(x,y)$ is the function of x and y; y is the solution to the differential equation in equation (1) at x_0 given as y_0; y_0 is the value of y obtained at x_0 and x_0 is the point for which y is obtained as y_0

$$y(x) = y_0 + \frac{y'(x-x_0)}{1!} + \frac{y''(x-x_0)^2}{2!} + \frac{y'''(x-x_0)^3}{3!} \tag{2}$$

Thus,

$$y(x_1) = y(x_0 + h) \tag{3}$$

$$y(x_0+h) = y(x_0) + \frac{hy'(x_0)}{1!} + \frac{h^2 y''(x_0)}{2!} + \frac{h^3 y'''(x_0)}{3!} + \ldots \frac{h^N y^N(x_0)}{N!} \tag{4}$$

Let $n = 1$

$$y(x_1) = y(x_0) + hy'(x_0) \tag{5}$$

equation (5) is the same as

$$y_1 = y_0 + hf(x_0, y_0) \tag{6}$$

$$y_2 = y_1 + hf(x_1, y_1) \tag{7}$$

$$y_3 = y_2 + hf(x_2, y_2) \tag{8a}$$

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{8b}$$

Equation (8b) is known as standard Euler method.

Modified Euler method

This method is a second order Runge-Kutta [5]. The convergence in this method is higher due to a higher degree of accuracy as compared to the standard Euler.

$$y_{n+1} = y_n + \frac{1}{2}hf\left[(x_n, y_n) + f(x_{n+1} + y_{n+1})\right] \tag{9}$$

DERIVATION:

Considering the Taylors series of $y(x_{n+1})$ about h given by

$$y(x_0 + h) = y(x_0) + \frac{hy_n^{'}}{1!} + \frac{h^2 y_n^{''}}{2!} + \frac{h^3 y_n^{'''}}{3!} + \dots \frac{h^N y_n^n}{N!} \tag{10}$$

Truncating when $n = 2$

$$y(x_n) = y_n + \frac{hy_n^{'}}{1!} + \frac{h^2 y_n^{''}}{2!} \tag{11}$$

From the definition of derivatives

$$y_n^{'} = f(x_n, y_n) \tag{12}$$

$$y_n^{''} \cong \frac{y^{'}h(x_n + h^*) - y^{'}(x_n)}{h^*} = f(x_n + h^*, y(x_n + h^*)) - \frac{f(x_n, y_n)}{h^*} \tag{13}$$

Thus,

$$y(x_n + h^*) = y_n + h^* f(x_n, y_n) \tag{14}$$

$$y_n^{''} \cong f(x_n + h^*, y(x_n + h^*)) - \frac{f(x_n, y_n)}{h^*} \tag{15}$$

Substitute $y_n^{''}$ and $y_n^{'}$ into equation (6):

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2\left(f(x_n + h^*, y(x_n + h^*)) - \frac{f(x_n, y_n)}{h^*}\right)} \tag{16}$$

Let

$$h^* = ah \tag{17}$$

Then we have

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2}\left(\frac{f(x_n + ah, y_n + ahf(x_n, y_n))}{ah}\right) \tag{18}$$

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h}{2a}\left(f(x_n + ah, y_n + ahf(x_n, y_n) - f(x_n, y_n))\right) \tag{19}$$

$$y_{n+1} = y_n + hf\left(x_n, y_n\right) + \frac{h}{2a}\left(f\left(x_n + ah, y_n + ahf\left(x_n, y_n\right)\right) - \frac{h}{2a} f\left(x_n, y_n\right)\right) \quad (20)$$

$$y_{n+1} = y_n + hf\left(x_n, y_n\right) - \frac{h}{2a} f\left(x_n, y_n\right) + \frac{h}{2a}\left(f\left(x_n + ah, y_n + ahf\left(x_n, y_n\right)\right)\right) \quad (21)$$

$$y_{n+1} = y_n + hf\left(x_n, y_n\right)\left(1 - \frac{h}{2a}\right) + \frac{h}{2a}\left(f\left(x_n + ah, y_n + ahf\left(x_n, y_n\right)\right)\right) \quad (22)$$

Equation (22) can be written as

$$y_{n+1} = y_n + c_1 k_1 + c_2 k_2 \quad (23)$$

where

$$c_1 = h\left(1 - \frac{1}{2a}\right) \quad (24)$$

$$c_2 = h\left(\frac{1}{2a}\right) \quad (25)$$

$$k_1 = f\left(x_n, y_n\right) \quad (26)$$

$$k_2 = f\left(x_n + ah, y_n + ahk_1\right) \quad (27)$$

Let

$$a = \frac{1}{2} \quad (28)$$

$$c_1 = h\left(1 - \frac{1}{2\frac{1}{2}}\right) = h(1-1) = 0 \quad (29)$$

$$c_2 = \left(\frac{h}{2\left(\frac{1}{2}\right)}\right) = h \quad (30)$$

$$k_1 = f\left(x_n, y_n\right) \quad (31)$$

$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \quad (32)$$

Hence,

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \quad (34)$$

OR

$$y_{n+1} = y_n + \frac{1}{2}hf\left[(x_n, y_n) + f\left(x_{n+1} + y_{n+1}\right)\right] \tag{35}$$

Runge-Kutta method

The Runge-Kutta method is also a second order Runge-Kutta Method using Taylors series expansion to derive it, like modified Euler's method [6].

From equation (22)

$$y_{n+1} = y_n + hf(x_n, y_n) \ \left(1 - \frac{h}{2a}\right) + \frac{h}{2a}\left(f\left(x_n + ah, y_n + ahf(x_n, y_n)\right)\right) \tag{36}$$

Equation (36) can be written as

$$y_{n+1} = y_n + c_1 k_1 + c_2 k_2 \tag{37}$$

where:

$$c_1 = h\left(1 - \frac{1}{2a}\right) \tag{38}$$

$$c_2 = \left(\frac{h}{2a}\right) \tag{39}$$

$$k_1 = f(x_n, y_n) \tag{40}$$

$$k_2 = f(x_n + ah, y_n + ahk_1) \tag{41}$$

Let

$$a = \frac{2}{3} \tag{42}$$

$$c_1 = h\left(1 - \frac{1}{2\frac{2}{3}}\right) = h\left(1 - \frac{3}{4}\right) = h\left(\frac{1}{4}\right) = \frac{1}{4}h \tag{43}$$

$$c_2 = \left(\frac{h}{2\left(\frac{2}{3}\right)}\right) = \frac{3h}{4} \tag{44}$$

$$k_1 = f(x_n, y_n) \tag{45}$$

$$\mathrm{k}_2 = \mathrm{f}\left(\mathrm{x}_n + \frac{2}{3}\mathrm{h}, \mathrm{y}_n + \frac{2}{3}\mathrm{hk}_1\right) \tag{46}$$

$$\Rightarrow y_{n+1} = y_n + \frac{1}{4}hf(x_n, y_n) + \frac{3h}{4}(f(x_n + ah, y_n + ahk_1)) \tag{47}$$

$$ie: y_{n+1} = y_n + \frac{1}{4}hk_1 + \frac{3h}{4}k_2 \tag{48}$$

$$thus: y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_2) \tag{49}$$

Third-stage Runge-Kutta method

The third-stage Runge-Kutta method express how formulations of *k* iterations are obtained.

$$ie: y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3) \tag{50}$$

where

$$k_1 = hf(x_0, y_0) \tag{51}$$

$$k_2 = hf(x_0 + h, y_0 + k_2) \quad k_4 = hf\left(x_9 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \tag{52}$$

Equation (52) is the third order Runge-Kutta method with error of order h^4.

Fourth-stage Runge-Kutta

One of the most frequently used of the Rung-Kutta family is the fourth order Runge-Kutta method or the classical fourth order Runge-Kutta method [7]. This method is generally superior to second order, its derivative is algebraically complicated and involves five equations.

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4) \tag{53}$$

where:

$$k_1 = hf(x_n, y_n) \tag{54}$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \tag{55}$$

$$k_4 = hf(x_n + h, y_n + hk_3) \tag{56}$$

Other renowned mathematicians that worked on this method are Runge-Kutta-Fehlberg and Runge-Kutta Nystrom.

Fehlberg's fifth – order R_k methodis:

$$y_{n+1} = y_n + \alpha_1 k_1 + \alpha_2 k_2 + \alpha_3 k_3 + \alpha_4 k_4 + \alpha_5 k_5 + \alpha_6 k_6 \quad (57)$$

where:

$$k_1 = hf(x_n, y_n) \quad (58)$$

$$k_2 = hf\left(x_n + \frac{1}{4}h, y_n + \frac{1}{4}hk_1\right) \quad (59)$$

$$k_3 = hf\left(x_n + \frac{3}{8}h, y_n + \frac{3}{32}hk_1 + \frac{9}{12}k_2\right) \quad (60)$$

$$k_4 = hf\left(x_n + \frac{12}{13}h, y_n + \frac{1932}{2197}hk_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \quad (61)$$

$$k_5 = hf\left(x_n + h, y_n + \frac{439}{2197}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \quad (62)$$

$$k_6 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{8}{27}hk_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4109}k_4 - \frac{11}{40}k_5\right) \quad (63)$$

A Finnish mathemathecian E.J. Nystron derived his own formular using the Runge-Kutta method [8].

$$k_1 = hf(x_n, y_n) \quad (64)$$

$$k_2 = hf\left(x_n + \frac{1}{4}h, y_n + \frac{1}{4}hk_1\right) \quad (65)$$

$$k_3 = hf\left(x_n + \frac{2}{3}h, y_n + \frac{1}{25}hk_1 + 6k_2\right) \quad (66)$$

$$k_4 = hf\left(x_n + \frac{3}{8}h, y_n + \frac{1}{4}hk_1 - 12k_2 + 15k_3\right) \quad (67)$$

$$k_5 = hf\left(x_n + \frac{2}{3}h, y_n + \frac{1}{81}h(6k_1) + 90k_2 - 50k_3 + 8k_4\right) \quad (68)$$

$$k_5 = hf\left(x_n + \frac{4}{5}h, y_n + \frac{1}{5}h(6k_1) + 36k_2 - 10k_3 + 8k_4\right) \quad (69)$$

Implementation of Euler method and Runge-Kutta method

The Euler and Runge-Kutta methods as previously deduced was used in this study to solve differential equations.

Standard Euler method

From equation (8), we have the formula

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{70}$$

Consider the differential equations in equations (71 and 86) of the initial value problem below:

$$y' = 3x^2 y; \quad y(0) = 1, h = 0.1, 0 \le x \le 1, \textit{Exact solution}: y = e^{x^3} \tag{71}$$

Solution:

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{72}$$

where

$$f(x_n, y_n) = y' = 3x_n^2 y_n; \quad y(0) = 1, h = 0.1, 0 \le x \le 1 \tag{73}$$

when $n = 0$

$$f(x_0, y_0) = 3x_9^2 y_0 = 3(0)^2(1) = 0$$

$$y = y_0 + hf(x_0, y_0) \tag{74}$$

$$= 1 + (0.1)(0) = 1.0000 \tag{75}$$

when $n = 0$

$$f(x_1, y_1) = 3x_1^2 y_1 = 3(0.1)^2(1) = 0.03$$

$$y_2 = y_1 + hf(x_1, y_1) \tag{76}$$

$$= 1 + (0.1)(0.03) = 1.00300 \tag{77}$$

when $n = 0$

$$f(x_2, y_2) = 3x_2^2 y_1 = 3(0.1)^2(1.00300) = 0.012036$$

$$y_3 = y_2 + hf(x_2, y_2) \tag{78}$$

$$= 1.00300 + (0.1)(0.12036)$$

$$= 1.00300 + (0.1)(0.12036) \tag{79}$$

$$y_4 = 1.04245, \; y_5 = 1.09249, \; y_6 = 1.09249, \; y_6 = 1.17443, \; y_7 = 1.30127 \tag{80}$$

$$y_8 = 1.49256, \; y_8 = 1.49256, \; y_9 = 1.77913, \; y_{10} = 2.14182, \; y_{11} = 2.78437 \tag{81}$$

Numerical illustration of exact solution for Example 1

The example below in equation (82) illustrates the application of exact solution for the solution of differential equations

$$y = e^{x^3},\ y_n = e^{x_n^3},\ \ 0 \le x \le 1 \tag{82}$$

If $n = 0$

$$y_0 = e^{x_0^3} = e^0 = 1 \tag{83}$$

when $n = 0$

$$y_1 = e^{x_1^3} = e^{(0.1)^3} = e^{(0.001)} = 1.00100 \tag{84}$$

when $n = 0$

$$y_2 = e^{x_2^3} = e^{(0.2)^3} = e^{(0.008)} = 1.00803 \tag{85}$$

$$\left.\begin{array}{l} y_3 = 1.02730,\ y_4 = 1.06609,\ y_5 = 1.13315,\ y_6 = 1.24110,\ y_7 = 1.40917, \\ y_8 = 1.66863,\ y_9 = 2.07301,\ y_{10} = 2.71828,\ y_{10} \end{array}\right\} \tag{86}$$

Consider the differential equation of the initial value problem:

$$y' = y - x;\quad y(0) = 2,\ h = 0.1,\ 0 \le x \le 1 \tag{87}$$

Exact solution

$$y = e^x + x + 1 \tag{88}$$

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{89}$$

where

$$f(x_n, y_n) = y' = y_n - x;\quad y_0 = 2 \tag{90}$$

when $n = 0$

$$f(x_0, y_0) = y_0 - x_0 = 2 - 0 = 2.00000$$

$$y_1 = y_0 + hf(x_0, y_0) = 2 + (0.1)(2.00000) = 2.20000 \tag{91}$$

when $n = 1$

$$f(x_1, y_1) = y_1 - x_1 = 2.20000 - 0.1 = 2.10000$$

$$y_2 = y_1 + hf(x_1, y_1) = 2.20000 + (0.1)(2.10000) = 2.41000 \tag{92}$$

when $n = 2$

$$f(x_2, y_2) = y_2 - x_2 = 2.41000 - 0.2 = 2.21000$$

$$y_2 = y_2 + hf(x_2, y_2) = 2.41000 + (0.1)(2.21000) = 2.61000 \tag{93}$$

$$y_3 = y_2 + hf(x_2, y_2) = 2.20000 + (0.1)(2.10000) = 2.41000 \tag{94}$$

$$y_4 = 2.86410,\ y_5 = 3.11051,\ y_6 = 3.37156,\ y_7 = 3.64872 \tag{95}$$

$$y_8 = 3.94360,\ y_9 = 4.25796,\ y_{10} = 4.59376,\ y_{11} = 4.95314 \tag{96}$$

Numerical illustration of exact solution for Example 2

Example 2 below in equation (97) is solved by finding the exact solution for the solution of differential equations

$$y = e^x + x + 1,\ 0 \le x \le 1 \tag{97}$$

$$y = e^{x_n} + x_n + 1 \tag{98}$$

when $n = 0$

$$y_0 = e^{x_n} + x_0 + 1 = e^0 + 0 + 1 = 1 + 1 = 2.00000 \tag{99}$$

when $n = 1$

$$y_1 = e^{x_1} + x_1 + 1 = e^{(0.1)} + 0.1 + 1 = 1.10517 + 1.1 = 2.20517 \tag{100}$$

when $n = 2$

$$y_2 = e^{x_2} + x_2 + 1 = e^{(0.2)} + 0.2 + 1 = 1.2 + 1.22140 + 1.2 = 2.42140 \tag{101}$$

$$y_3 = 2.64986,\ y_4 = 2.89182,\ y_5 = 3.14872,\ y_6 = 3.42212 \tag{102}$$

$$y_7 = 371375,\ y_8 = 4.02554,\ y_9 = 4.35960,\ y_7 = 4.71828 \tag{103}$$

Modified Euler method

This is the second Euler's method we are considering as deduced in equations (23-35) and expressed as:

$$y_{n+1} = y_n + hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \tag{104}$$

where

$$k_1 = f(x_n, y_n),\ k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \tag{105}$$

$$y_{n+1} = y_n + hk_2 \tag{106}$$

Let us consider the same differential equations in equations (72 and 87):

$$y' = 3x^2 y; \quad y(0) = 1,\ h = 0.1,\ 0 \le x \le 1,\ Exact\ solution : y = e^{x^3} \tag{107}$$

Solution

$$y_{n+1} = y_n + hk_2,\ f(x_n, y_n) = 3x_n^2 y_n \tag{108}$$

Issue 32, January-June 2018
p. 10-37

when $n = 0$

$$k_1 = f(x_0, y_0) = 3x_0^2 y_0 = 3(0)^2(1) = 0.00000$$

$$k_2 = f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hk_1\right) = f\left(0 + \frac{1}{2}(0.1). \frac{1}{2(0.1)(0.00000)}\right) = f(0.05,1)$$

$$k_2 = f(0.05,1) = 3(0.05)^2(1) = 0.00750$$

$$y_{n+1} = y_n + hk_2 = 1 + (0.1)(0.00750) = 1 + 0.00075 = 1.00075 \quad (109)$$

when $n = 1$

$$k_1 = f(x_1, y_1) = 3x_1^2 y_1 = 3(0.1)^2(1.00075) = 0.03002$$

when $n = 2$

$$k_2 = f\left(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hk_1\right) = f\left(0.1 + \frac{1}{2}(0.1), (1.00075)\right) = f(0.15,1.00251)$$

$$k_2 = f(0.15,1.00251) = 3(0.15)^2(1.00251) = 0.06753$$

$$y_2 = y_1 + hk_2 = 1.00075 + (0.1)(0.06753) = 1.00750 \quad (110)$$

when $n = 2$

$$k_1 = f(x_2, y_2) = 3x_2^2 y_2 = 3(0.2)^2(1.00750) = 0.12090$$

$$k_2 = f\ \ x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}hk_1\ \ =$$

$$= f\ \ 0.2 + \frac{1}{2}(0.1), (1.075) + \frac{1}{2(0.1)(0.12090)}\ \ =$$

$$= f(0.15,1.00251)$$

$$k_2 = f(0.25,1.01355) = 3(0.25)^2(1.01255) = 0.76016$$

$$y_2 = y_2 + hk_2 = 1.00075 + (0.1)(0.76016) = 1.08352 \quad (111)$$

$$y_4 = 1.12076,\ y_5 = 1.19050,\ y_6 1.30259,\ y_7 = 1.47661$$

$$y_8 = 1.74410,\ y_9 = 2.15842,\ y_{10} = 2.81384,\ y_{11} = 3.88412$$

Numerical illustration of Exact solution for Example 1

Applying exact solution technique to show the solution of the problem in example 1

$$y = e^{x^3} \quad (112)$$

$$y = e^{x^3},\ y_n = e^{x_n^3},\ 0 \le x \le 1 \quad (113)$$

when $n = 0$, $y_0 = e^{x_0^3} = e^0 = 1$

when $n = 1$

$$y_1 = e^{x_1^3} = e^{(0.1)^3} = e^{(0.001)} = 1.00100 \quad (114)$$

when $n = 1$

$$y_1 = e^{x_2^3} = e^{(0.2)^3} = e^{(0.008)} = 1.00803 \quad (115)$$

$y_3 = 1.02730,\ y_4 = 1.06609,\ y_5 = 1.13315,\ y_6 = 1.24110,\ y_7 = 1.40917,$
$y_8 = 1.66863,\ y_9 = 2.07301,\ y_{10} = 2.71828$

Consider the differential equation:

$$y' = y - x; \quad y(0) = 2,\ h = 0.1,\ 0 \le x \le 1 \quad (116)$$

Solution

$$y_{n+1} = y_n + hk_2 \quad (117$$

$$k_1 = f(x_n, y_n) = y_n - x_n,\ k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right) \quad (118)$$

when $n = 0$

$$k_1 = f(x_0, y_0) = y_0 - x_0 = 2 - 0 = 2$$

$$k_2 = f\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hk_1\right) = f\left(0 + \frac{1}{2}(0.1), 2 + \frac{1}{2(0.1)(2)}\right) = f(0.05, 2.1) \quad (119)$$

$$= y_n - x_n = 2.1 - 0.05 = 2.05$$

$$y_{n+1} = y_0 + hk_2 = 2 + (0.1)(2.05) = 2.20500 \quad (120)$$

when $n = 1$

$$k_1 = f(x_1, y_1) = y_1 - x_1 = 20500 - 0.1 = 2.10500 \quad (121)$$

$$k_2 = f\ x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}hk_1 = f\ 0.1 + \frac{1}{2}(0.1), 2.20500 - \frac{1}{2(0.1)(2.20500)} = f(0.15, 2.3102) \quad (122)$$

$$= y_n\ x_n = 2.310250.15 = 2.16025$$

$$y_2 = y_1 + hk_2 = 2.20500 + (0.1)(2.16025) = 2.42103 \quad (123)$$

when $n = 2$

$$k_1 = f(x_2, y_2) = y_2 - x_2 = 2.42103 - 0.2 = 2.22103 \quad (124)$$

$$k_2 = f\left(x_2 + \frac{1}{2}h, y_2 + \frac{1}{2}hk_1\right) = f\left(0.2 + \frac{1}{2}(0.1), 2.42103 - \frac{1}{2(0.1)(2.22103)}\right) = f(0.25, 2.5320\ \quad (125)$$

$$= y_n - x_n = 2.53208 - 0.25 = 2.28208$$

$$y_2 = y_0 + hk_2 = 2 + (0.1)(2.28208) = 2.62821 \quad (126)$$

$$y_4 = 2.89091,\ y_5 3.15991,\ y_6 = 3.4320,\ y_7 = 3.72671,$$
$$y_8 = 4.03960,\ y_9 = 4.37476,\ y_{10} = 5.12147$$

Numerical illustration of exact solution for Example 2

Using exact solution approach to show the solution of the problem in example 2.

$$y = e^x + x + 1,\ 0 \le x \le 1 \quad (127)$$

$$y = e^{x_n} + x_n + 1 \quad (128)$$

when $n = 0$

$$y_0 = e^{x_0} + x_0 + 1 = e^{(0)} + 0 + 1 = 1 + 12.00000 \quad (129)$$

when $n = 1$

$$y_1 = e^{x_1} + x_1 + 1 = e^{(0).1} + 0.1 + 1 = 1.10517 + 1.1 = 2.20517 \quad (130)$$

when $n = 2$

$$y_2 = e^{x_2} + x_2 + 1 = e^{(0).2} + 0.2 + 1 = 1.22140 + 1.2 = 2.42140 \quad (131)$$

$$y_3 = 2.64986,\ y_4 = 2.89182,\ y_5 = 3.14872\ y_6 = 3.42212,\ y_7 = 3.71375,$$
$$y_8 = 4.02554,\ y_9 = 4.35960,\ y_{10} = 4.71828$$

Runge-Kutta method

From equation (49), Heun's method formula is thus given as:

$$y_{n+1} = y_n + \frac{1}{4}(k_1 + 3k_2) \quad (132)$$

where:

$$k_1 = f(x_n, y_n),\ k_2 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1\right) \quad (133)$$

Consider the differential equation of the initial value problem.

$$y = 3x^2 y;\ y(0) = 1, h = 0.1,\ 0 \le x \le 1 \quad (134)$$

Exact solution

$$y = e^{x^3} \quad (135)$$

Solution

$$f(x_n, y_n) = 3x_n^2 y_n \quad (136)$$

when $n = 0$

$$k_1 = f(x_0, y_0) = 3x_0^2 y_0 = 3(0)^2(1) = 0 \tag{137}$$

$$k_2 = f\left(x_0 + \frac{2}{3}h, y_0 + \frac{2}{3}hk_1\right) = f\left(0 + \frac{2}{3}(0.1), 1 + \frac{1}{3(0.1)(0)}\right) \tag{138}$$

$$k_1 = f(0.66667, 1) = 3(0.66667)^2(1) = 0.01335 \tag{139}$$

$$y_1 = y_0 + \frac{h}{4} f(k_1 + 3k_2 + 3k_2) = 1 + \frac{0.1}{4}(0 + 3(0.01335)) = 1.0010 \tag{140}$$

when $n = 1$

$$k_1 = f(x_1, y_1) = 3x_1^2 y_1 = 3(0.1)^2(1.00100) = 0.03003 \tag{141}$$

$$k_2 = f\left(x_1 + \frac{2}{3}h, y_1 + \frac{2}{3}hk_1\right) = f\left(0.1 + \frac{2}{3}(0.1), 1.00100 + \frac{2}{3(0.1)(0.03003)}\right) \tag{142}$$

$$k_1 = f(0.16667, 1.00300) = 3(0.16667)^2(1.00300) = 0.08359 \tag{143}$$

$$y_2 = y_1 + \frac{h}{4} f(k_1 + 3k_2) = 1.00100 + \frac{0.1}{4}(0.003003 + 3(0.08359)) \tag{144}$$

$$y_3 = 1.00100 + 0.00702 = 1.00802 \tag{145}$$

When $n = 2$

$$k_1 = f(x_2, y_2) = 3x_2^2 y_1 = 3(0.2)^2(1.00802) = 0.12096 \tag{146}$$

$$k_2 = f\left(x_2 + \frac{2}{3}h, y_2 + \frac{2}{3}hk_1\right) = f\left(0.2 + \frac{2}{3}(0.1), 1.00802 + \frac{2}{3(0.1)(0.12096)}\right) \tag{147}$$

$$k_1 = f(0.26667, 1.01608) = 3(0.26667)^2(1.01608) = 0.21677 \tag{148}$$

$$y_2 = y_2 + \frac{h}{4} f(k_1 + 3k_2) = 1.00802 + \frac{0.1}{4}(0.12096 + 3(0.21677)) \tag{149}$$

$$y_3 = 1.00802 + 0.01928 = 1.02730 \tag{150}$$

$y_4 = 1.06587$, $y_5 = 1.09345$ $y_6 = 1.19691$, $y_7 = 1.35754$,
$y_8 = 1.60456$, $y_9 = 1.98746$, $y_{10} = 2.59376$, $y_{11} = 3.58510$

Numerical illustration of exact solution for Example 1

$$y = e^{x^3} \tag{151}$$

$$y = e^{x^3},\ y_n = e^{x_n^3},\ 0 \le x \le 1 \tag{152}$$

when $n = 0$

$$y_0 = e^{x_0^3} = e^0 = 1 \qquad (153)$$

when $n = 1$

$$y_1 = e^{x_1^3} = e^{(0.1)^3} = e^{(0.001)} = 1.00100 \qquad (154)$$

when $n = 2$

$$y_1 = e^{x_2^3} = e^{(0.2)^3} = e^{(0.008)} = 1.00803 \qquad (155)$$

$y_3 = 1.02730,\ y_4 = 1.06609\ y_5 = 1.13315,\ y_6 = 1.24110,$
$y_7 = 1.40917,\ y_8 = 1.66863,\ y_9 = 2.07301,\ y_{10} = 2.71828$

Consider the differential equation of the initial value problem:

$$y' = y - x; \quad y(0) = 2,\ h = 0.1,\ 0 \le x \le 1 \qquad (156)$$

Exact solution

$$y = e^{x^3} + x + 1 \qquad (158)$$

Solution:

$$y_{n+1} = y_n + \frac{h}{4}(k_2 + 3k_2) \qquad (159)$$

$$k_1 = f(x_n, y_n) y' = y_n - x_n,\ k_2 = f\left(x_n + \frac{2}{3}h, y_n + \frac{2}{3}hk_1\right) \qquad (160)$$

when $n = 0$

$$k_1 = f(x_0, y_0) = y_0 - x_0 = 2 - 0 = 2 \qquad (161)$$

$$k_2 = f\left(x_0 + \frac{2}{3}h, y_0 + \frac{2}{3}hk_1\right) = f\left(0 + \frac{2}{3}(0.1), 2 + \frac{2}{3(0.1)(2)}\right) \qquad (162)$$

$$k_1 = f(0.06667, 2.13333) = y_n - x_n = 2.13333 - 0.06667 = 2.06667 \qquad (163)$$

$$k_1 = f(0.06667, 2.13333) = y_n - x_n = 2.13333 - 0.06667 = 2.06667 \qquad (164)$$

when $n = 1$

$$k_1 = f(x_1, y_1) = y_1 - x_1 = 1.77000 - 0.1 = 1.67000 \qquad (165)$$

$$k_2 = f\left(x_1 + \frac{2}{3}h, y_1 + \frac{2}{3}hk_1\right) = f\left(0.1 + \frac{2}{3}(0.1), 1.77000 + \frac{2}{3(0.1)(1.7700)}\right) \qquad (166)$$

$$k_1 = f(1.16667, 1.88133) = y_n - x_n = 1.88133 - 1.16667 = 2.71466 \qquad (167)$$

$$y_1 = y_1 + \frac{h}{4}(k_1 + 3k_2) = 1.77000 + \frac{0.1}{4}(1.67000 + 3(1.71466)) = 1.9403 \qquad (168)$$

when $n = 2$

$$k_1 = f(x_2, y_2) = y_2 - x_2 = 1.94035 - 0.2 = 1.74035 \qquad (169)$$

$$k_2 = f\left(x_2 + \frac{2}{3}h, y_2 + \frac{2}{3}hk_1\right) = f\left(0.2 + \frac{2}{3}(0.1), 1.94035 + \frac{2}{3(0.1)(1.74035)}\right) \qquad (170)$$

$$k_1 = f(0.26667, 2.05637) = y_n - x_n = 2.05637 - 0.26667 = 2.78970 \qquad (171)$$

$$y_2 = y_2 + \frac{h}{4}(k_1 + 3k_2) = 1.94035 + \frac{0.1}{4}(1.74035 + 3(1.71466)) = 2.1180 \qquad (172)$$

$$y_4 = 2.80399 y_5 = 2.49891,\ y_6 = 2.70380,\ \ y_7 = 2.91970,$$
$$y_8 = 3.14777,\ y_9 = 23.38929,\ y_{10} = 3.64567,\ y_{11} = 3.92094$$

Numerical illustration of exact solution for Example 2

$$y = e^{x^3} + x + 1 \qquad (173)$$

$$y = e^{x_n} + x_n + 1 \qquad (174)$$

when $n = 0$

$$y_0 = e^{x_0} + x_0 + 1 = e^0 + 0 + 1 = 1 + 1 = 2.0000 \qquad (175)$$

when $n = 1$

$$y_0 = e^{x_1} + x_1 + 1 = e^{(0.1)} + 0.1 + 1 = 1.10517 + 1.1 = 2.20517 \qquad (176)$$

when $n = 1$

$$y_2 = e^{x_2} + x_2 + 1 = e^{(0.2)} + 0.2 + 1 = 1.22140 + 1.2 = 2.42140 \qquad (177)$$

$$y_3 = 2.64986,\ y_4 = 2.89182\ y_5 = 3.14872,\ y_6 = 3.42212,\ \ y_7 = 3.71375, \qquad (178)$$

$$y_8 = 4.02554,\ y_9 = 4.35960, y_{10} = 4.71828 \qquad (179)$$

Analysis of Euler and Runge-Kutta methods

The Euler and Runge-Kutta methods were compared by a rigorous analysis in order to demonstrate the efficiency of the methods to other techniques. The effect of the steps on the accuracy of the techniques was also examined.

The results obtained by solving some differential equations as considered in this paper were analyzed based on the theory of comparison. This help towards bringing out the result clearly for easy analysis and understanding of the established concept.

Results and Discussion

Standard Euler method

Table 1 showed that the results improved greatly when the standard Euler's method was used. The tables below show the summary of all the results for the methods in consideration.

In comparing the two methods, we can see clearly that the Euler method is more preferable than the Runge-Kutta, especially when many values of x_n are required. If only a few values of x_n are needed, then the Runge-Kutta techniques is preferred to Euler method because it gives slightly better results

Table 1. Solution for $y' = 3x^2 y; y(0) = 1, h = 0.1, 0 \le x \le 1$

N	H	$f(x_n, y_n)$	y_{n+1}	Exact solution	Error
0	0.1	0	1.00000	1	0
1	0.1	0.03	1.00300	1.00100	0.002
2	0.1	0.12036	1.01504	1.00803	0.00701
3	0.1	0.27406	1.04245	1.02730	0.01515
4	0.1	0.50038	1.09245	1.06609	0.02636
5	0.1	0.81938	1.17443	1.13315	0.04128
6	0.1	1.26838	1.30127	1.24110	0.06017
7	0.1	1.91287	1.49256	1.40917	0.08339
8	0.1	2.86572	1.77913	1.66863	0.1105
9	0.1	3.62692	2.14182	2.07301	0.06881
10	0.1	6.42546	2.78437	2.71828	0.06609

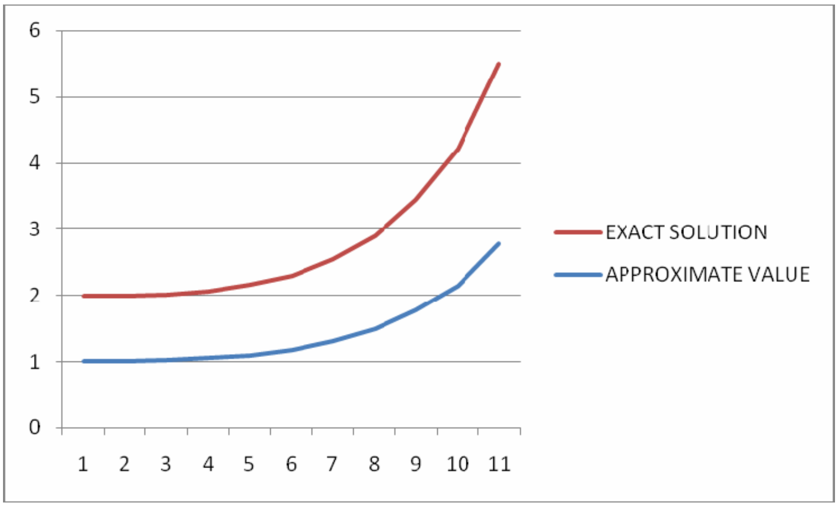

Figure 1. Graph of Approximate solution of standard Euler compare to exact solution

Table 2. Solution for $y' = y - x; \quad y(0) = 2, h = 0.1, 0 \le x \le 1$

N	H	$f(x_n, y_n)$	y_{n+1}	Exact number	Error
0	0.1	2.00000	2.20000	2.0000	0.2
1	0.1	2.10000	2.41000	2.20517	0.20483
2	0.1	2.21000	2.61000	2.42140	0.1886
3	0.1	2.33100	2.86410	2.64986	0.21424
4	0.1	2.46410	3.11051	2.89182	0.21869
5	0.1	2.61051	3.37156	3.14872	0.22284
6	0.1	2.77156	3.64872	3.42212	0.2266
7	0.1	2.94872	3.94360	3.71375	0.22985
8	0.1	3.14360	4.25796	4.02554	0.23242
9	0.1	3.35796	4.59376	4.35960	0.23416
10	0.1	3.56376	4.95314	4.71828	0.23486

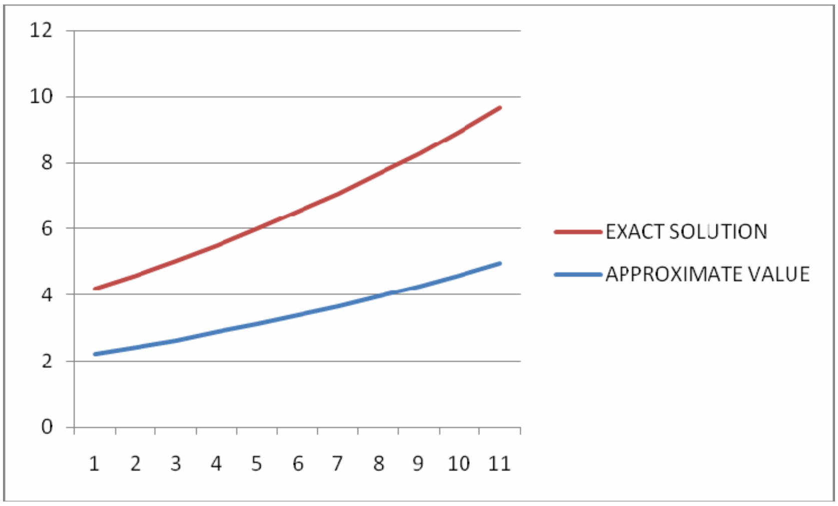

Figure 2. Graph of Approximate solution of standard Euler compared to exact solution

Modified Euler Method

The performance of the modified Euler method is presented in Table 3 and 4 and graphically depicted in Figure 3 and 4.

Table 3. Solution for $y' = 3x^2 y; y(0) = 1, h = 0.1, 0 \le x \le 1$

N	H	$f(x_n, y_n)$	y_{n+1}	Exact number	Error
0	0.1	0.00000	1.00075	2.0000	0.99925
1	0.1	0.03002	1.00750	2.20517	1.19767
2	0.1	0.12090	1.08352	2.42140	1.33788
3	0.1	0.29255	1.12076	2.64986	1.5291
4	0.1	0.53769	1.19050	2.89182	1.70132
5	0.1	0.89288	1.30259	3.14872	1.84613
6	0.1	1.40680	1.47661	3.42212	1.94551
7	0.1	2.17062	1.74410	3.71375	1.96965
8	0.1	3.34867	2.15842	4.02554	1.86712
9	0.1	5.24496	2.81384	4.35960	1.54576
10	0.1	8.44151	3.88412	4.71828	0.83416

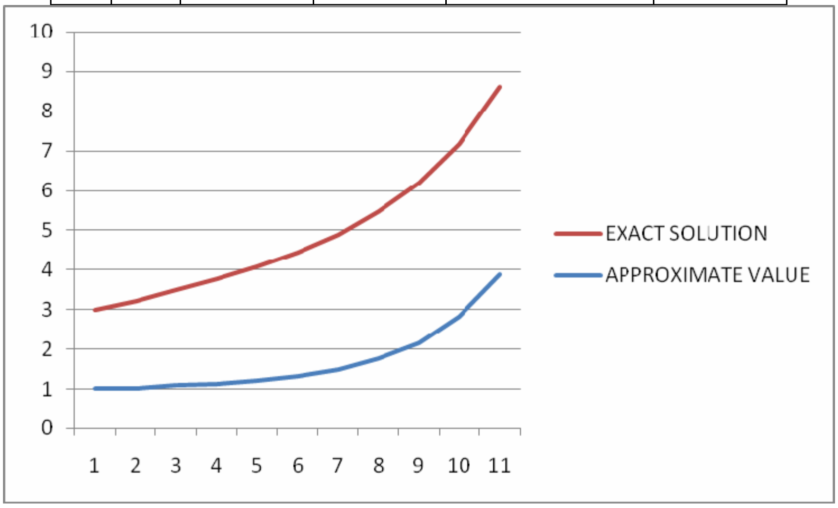

Figure 3. Graph of Approximate solution of Modified Euler compared to exact

Table 4. Solution for $y' = y - x; \quad y(0) = 2, h = 0.1, 0 \le x \le 1$

N	H	$f(x_n, y_n)$	y_{n+1}	Exact number	Error
0	0.1	2	2.20500	1	1.205
1	0.1	2.10500	2.42103	1.00100	1.42003
2	0.1	2.22103	2.62821	1.00803	1.62018
3	0.1	2.34924	2.89091	1.02730	1.86361
4	0.1	2.49091	3.15991	1.06609	2.09382
5	0.1	2.65991	3.43420	1.13315	2.30105
6	0.1	2.83420	3.72671	1.24110	2.48561
7	0.1	3.02679	4.03960	1.40917	2.63043
8	0.1	3.23960	4.37476	1.66863	2.70613
9	0.1	3.47476	4.73461	2.07301	2.6616
10	0.1	3.73461	5.12147	2.71828	2.40319

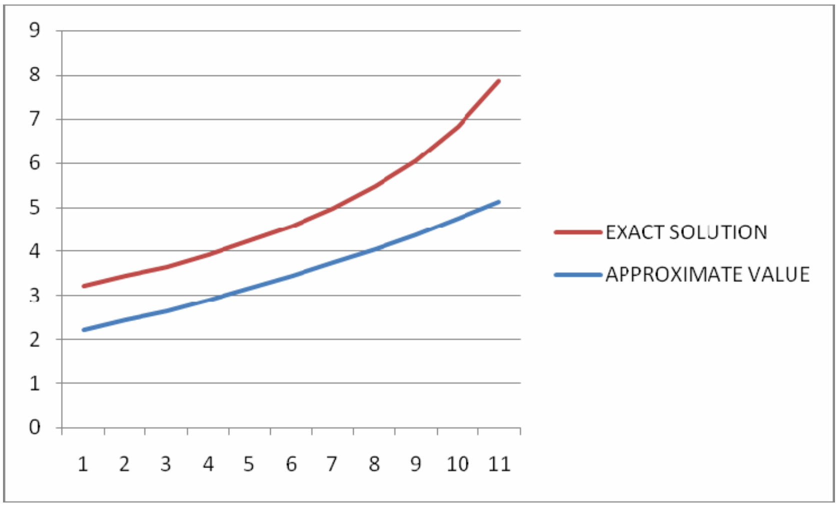

Figure 4. Graph of Approximate solution of Modified Euler compared to exact solution

Runge-Kutta method

The performances of the Runge-Kutta method are presented in Table 5 and 6 and graphically depicted in Figure 5 and 6.

Table 5. Solution for $y' = 3x^2 y; y(0) = 1, h = 0.1, 0 \le x \le 1$

N	H	$f(x_n, y_n)$	y_{n+1}	Exact number	Error
0	0.1	0	1.00100	1	0.001
1	0.1	0.3003	1.00802	1.00100	0.00702
2	0.1	0.12096	1.02730	1.00803	0.01927
3	0.1	0.27737	1.06587	1.02730	0.03857
4	0.1	0.55162	1.09345	1.06609	0.02736
5	0.1	0.82009	1.19691	1.13315	0.06376
6	0.1	1.29266	1.35754	1.24110	0.11644
7	0.1	1.99558	1.60456	1.40917	0.19539
8	0.1	3.08076	1.98746	1.66863	0.31883
9	0.1	4.82953	2.59376	2.07301	0.52075
10	0.1	7.78129	3.58510	2.71828	0.86682

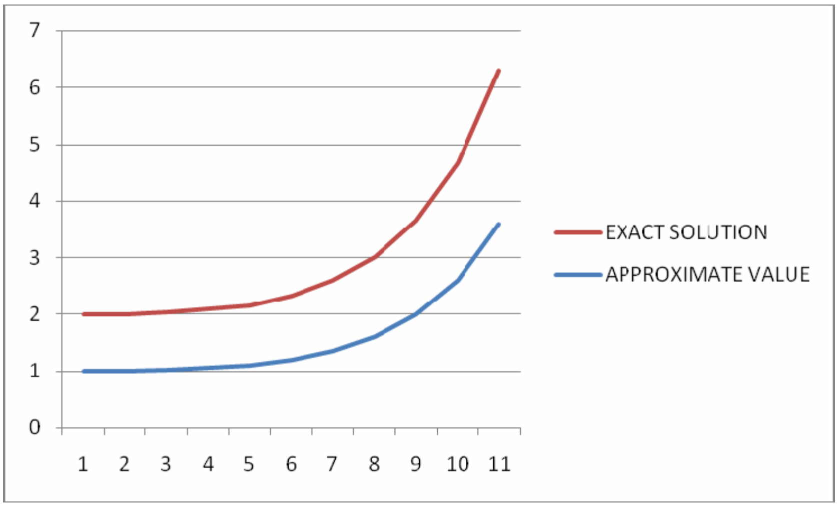

Figure 5. Graph of Approximate solution of Runge-kutta compared to exact solution

Table 6. Solution for $y^{'} = y - x; \quad y(0) = 2, h = 0.1, 0 \le x \le 1$

N	H	$f(x_n, y_n)$	y_{n+1}	Exact number	Error
0	0.1	1	1.7700	2.0000	0.23
1	0.1	1.67000	1.94035	2.20517	0.26482
2	0.1	1.74035	2.11809	2.42140	0.30331
3	0.1	1.81809	2.80399	2.64986	0.15413
4	0.1	1.90399	2.49891	2.89182	0.39291
5	0.1	1.99891	2.70380	3.14872	0.44492
6	0.1	2.10380	2.91970	3.42212	0.50242
7	0.1	2.21970	3.14777	3.71375	0.56598
8	0.1	2.34777	3.38929	4.02554	0.63625
9	0.1	2.48929	3.645667	4.35960	0.71393
10	0.1	2.64567	3.92094	4.71828	0.79734

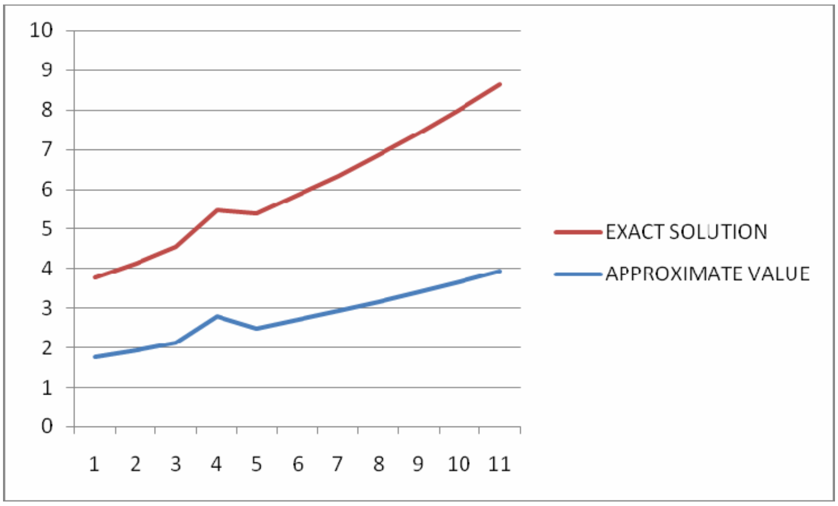

Figure 6. Graph of approximate solution of Runge-kutta compared to exact solution

Methods comparison

The performances of the methods for specific scenario are presented in Table 7 to 10 and graphically represented in Figure 7 to 10.

Table 7. Solution for $y' = 3x^2 y; y(0) = 1, h = 0.1, 0 \le x \le 1$

Runge-Kutta	Standard Euler	Modified Euler	Exact Solution
1.001	1	1.00075	1
1.00802	1.003	1.0075	1.001
1.0273	1.01504	1.08352	1.00803
1.06587	1.04245	1.12076	1.0273
1.09345	1.09245	1.1905	1.06609
1.19691	1.17443	1.30259	1.13315
1.35754	1.30127	1.47661	1.2411
1.60456	1.49256	1.7441	1.40917
1.98746	1.77913	2.15842	1.66863
2.59376	2.14182	2.81384	2.07301
3.5851	2.78437	3.88412	2.71828

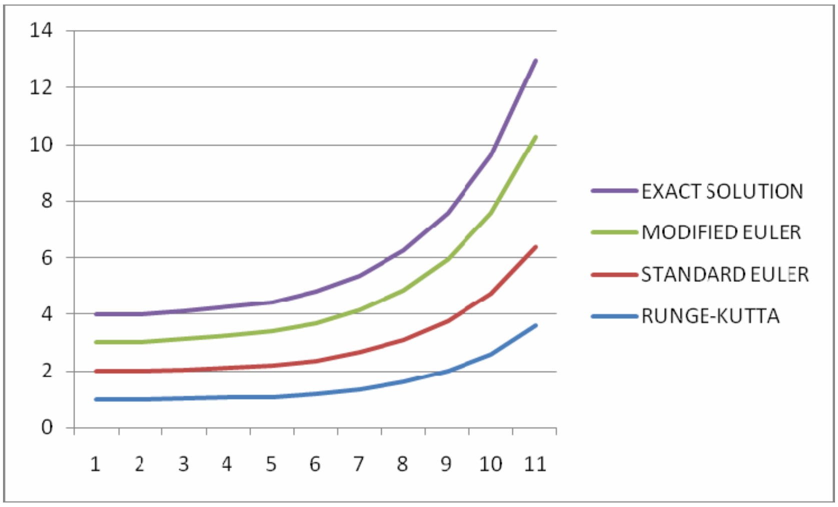

Figure 7. Comparison of Exact Solution, Standard Euler, Modified Euler and Runge-Kutta methods

Table 6. Solution for $y^{'} = y - x; \quad y(0) = 2,\ h = 0.1,\ 0 \le x \le 1$

Runge-Kutta	Standard Euler	Modified Euler	Exact Solution
1.77	2.2	2.205	1
1.94035	2.41	2.42103	1.001
2.11809	2.61	2.62821	1.00803
2.80399	2.8641	2.89091	1.0273
2.49891	3.11051	3.15991	1.06609
2.7038	3.37156	3.4342	1.13315
2.9197	3.64872	3.72671	1.2411
3.14777	3.9436	4.0396	1.40917
3.38929	4.25796	4.37476	1.66863
3.645667	4.59376	4.73461	2.07301
3.92094	4.95314	5.12147	2.71828

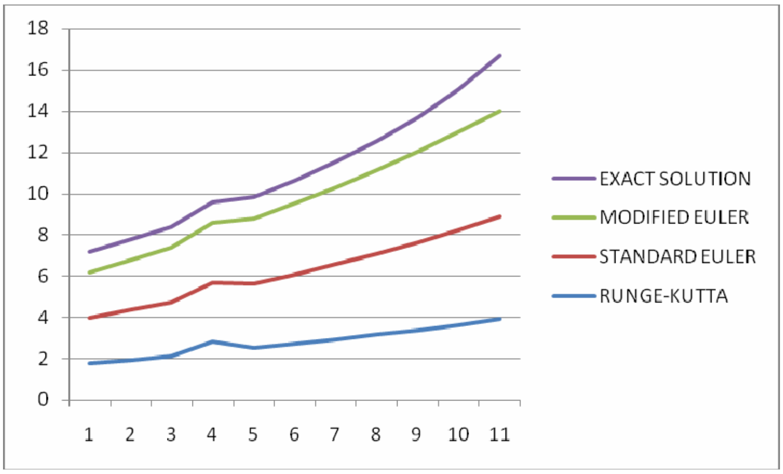

Figure 8. Comparison of Exact Solution, Standard Euler, Modified Euler and Runge-Kutta methods

Table 9. Absolute Errors for $y^{'} = 3x^2 y; y(0) = 1, h = 0.1,\ 0 \le x \le 1$

Runge-Kutta	Standard Euler	Modified Euler
0.001	0	0.99925
0.00702	0.002	1.19767
0.01927	0.00701	1.33788
0.03857	0.01515	1.5291
0.02736	0.02636	1.70132
0.06376	0.04128	1.84613
0.11644	0.06017	1.94551
0.19539	0.08339	1.96965
0.31883	0.1105	1.86712
0.52075	0.06881	1.54576
0.86682	0.06609	0.83416

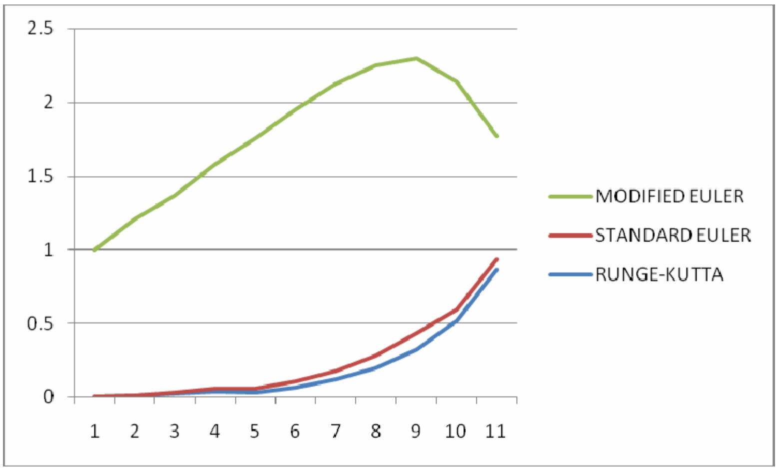

Figure 9. Graph of Absolute Errors for Standard Euler, Modified Euler and Runge-Kutta methods for $y' = 3x^2 y; y(0) = 1, h = 0.1, 0 \le x \le 1$

Table 10. Absolute error for $y' = y - x; \quad y(0) = 2, h = 0.1, 0 \le x \le 1$

Runge-Kutta	**Standard Euler**	**Modified Euler**
0.23	0.2	1.205
0.26482	0.20483	1.42003
0.30331	0.1886	1.62018
0.15413	0.21424	1.86361
0.39291	0.21869	2.09382
0.44492	0.22284	2.30105
0.50242	0.2266	2.48561
0.56598	0.22985	2.63043
0.63625	0.23242	2.70613
0.71393	0.23416	2.6616
0.79734	0.23486	2.40319

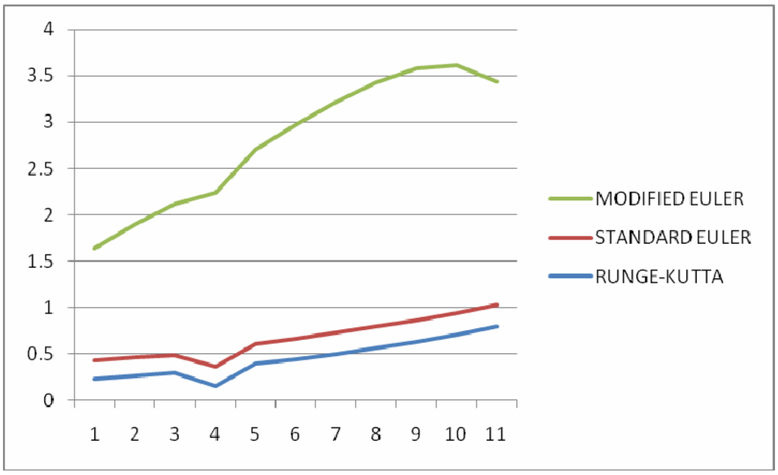

Figure 10. Graph of absolute errors for Standard Euler, Modified Euler and Runge-Kutta methods for $y' = y - x; \quad y(0) = 2, h = 0.1, 0 \le x \le 1$

This research work has been carried out in order to compute direct solution of numerical method for order four of ordinary differential equations without reducing it to the system of first order differential equation through the application of both Euler and Runge-Kutta methods. These methods sufficiently reduce computational efforts as well as provide the accuracy needed in each result. It was analyzed from figures (1-10) and the percentage errors as (0.991% and 4.902% from Tables 1 and 2 for standard Euler method), (33.360% and 46.477% from Tables 3 and 4 for Modified Euler method) and (4.322% and 9.948% from Tables 5 and 6 for Runge-kutta method) which became necessary for a reasonable conclusion to be made.

Generally, the aim of a good numerical differentiation is that such methods should give better approximations to the true differentials. By the examples above, it shows that the two methods produced better degrees of accuracy for ordinary differential equations of all orders. But comparing the percentage errors, we discovered that Standard Euler method is more accurate than Modified Euler [3, 10, 11] and Runge-Kutta methods for the examples illustrated.

The exact solutions for both Euler and Runge-Kutta methods, whose values have the same order of accuracy with the derived solutions, were not only formulated, also tested for simplicity, efficiency and accuracy. We can conclude from the derivations above that the Standard Euler's method has a higher degree of accuracy than the Modified Euler's method, conclusion also supported by other studies [3, 10-15]. It is remarkable therefore to note that all the examples illustrated are differential equations of order four and orders which are less than four. This, to a large extent, reduces the effect that global error could have on the accuracy of the methods as obtained in tables (9 and 10) as well as the corresponding graphs in figures (9 and 10) respectively.

The methods were also examined to solve both linear and non-linear problems of the fourth order differential equations. Hence by percentage error, standard Euler method is ranked more accurate than Runge-kutta method with (0.991% and 4.902%), (4.322% and 9.948%) for the results in tables (1 and 2), (5 and 6) respectively.

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